Iterations

By David J. Staley

Gentlemen, I thank you for the invitation to present my first essay to the Kit Kat club. When I was invited to join this august group, I was drawn first to the quality of its membership, and to its commitment to thoughtful, intellectual discourse. What intrigued me most was the essay, and the idea that the subject of that essay is to be in an area outside of the gentlemen's expertise. I had found a group of like-minded dilatants (which I do not consider an insult but, as it was in the 18th century, a term of high praise!) Because I am an historian by profession, I suppose this bars me from making a presentation about history. I note that so many of you, when giving your essays, usually light upon historical topics; there are so many "history buffs" in this group! As it turns out, I am a "mathematics buff," and so tonight, I and would like to explore with you one of my favorite mathematical objects.

Let me first begin with an equation, which I promise is the only one you will encounter this evening, for indeed it is the only one we need:

$$z = z^2 + C$$

What this means is that we will take a number (z), square that number and then add a constant. That's it. This is a simple enough operation, and would not seem to be very interesting at all. But this equation is what mathematicians call an "iterated function," which means that the result we get from squaring a number and adding a constant is then used as the new value for z. We then take that new number, square it, add the same constant we used before and get a new result, which we then plug back in as the value for z and repeat this process, *ad infinitum*. Iterated functions have some practical uses, such as studying the dynamics of population growth. But my interest in these equations comes from the interesting patterns that are generated.

You see, with iterated functions of this type, it is not the *results* that are particularly interesting. What draws our attention are the *behaviors* of the dynamical system (the dynamism here being the iterative process.) What happens when we plug in a number into this function and iterate it lots of times? What interesting things happen?

Here's a simple example of what I mean. Let's take our equation and plug in some numbers. Let's start first with 0, and plug that number in as z. Let's set our constant C as 1. We then get the following result:

$Z_0 = 0^2 + 1 = 0 + 1 = 1$

We then take the 1 and use this as the new value for z and calculate the result again:

$Z_1 = 1^2 + 1 = 1 + 1 = 2$

(By the way, the z-sub number refers to the number of iterations we have conducted. So z_2 means that this is the second iteration.) Then we take the 2 and do the same calculation again:

$Z_2 = 2^2 + 1 = 4 + 1 = 5$

And again:

$Z_3 = 5^2 + 1 = 25 + 1 = 26$

And again:

$Z_4 = 26^2 + 1 = 676 + 1 = 677$

And again:

$Z_5 = 677^2 + 1 = 458329 + 1 = 458330$

And so on.

What have we learned here? Again, with iterated functions we are less interested in the results as we are in the "behavior" of the system, in this case, the behavior that this equation exhibits when $z = 0$ and when our constant is 1. We learn that very quickly, after only five iterations, our results are heading off toward infinity. The next iteration, were I to actually calculate it, would have been an even larger number, and the one after that even larger. Clearly, this equation, under these starting conditions, is heading toward infinity.

Now let's plug in different numbers and see what happens. Let's keep $z = 0$ but this time let's set our constant at 0 as well. It should be apparent that such an iteration would always yield the result 0, since:

$Z_0 = 0^2 + 0 = 0 + 0 = 0$

$Z_1 = 0^2 + 0 = 0 + 0 = 0$

$Z_2 = 0^2 + 0 = 0 + 0 = 0$

Mathematicians would say that, under these conditions, our results "orbit at 0," meaning that our iterated results will always settle around that number.

Now let's set our constant at -1. Here's what happens:

$Z_0 = 0^2 - 1 = 0 - 1 = -1$

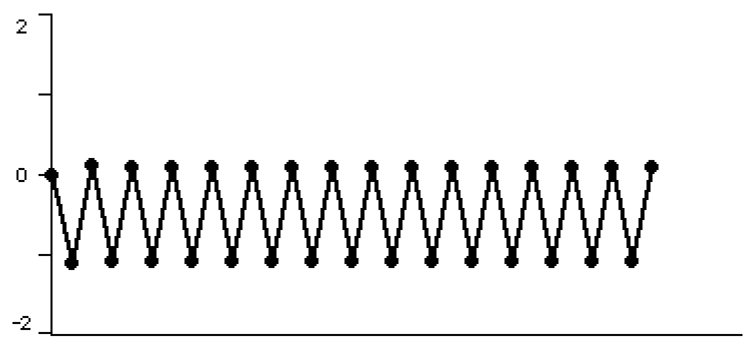
$Z_1 = -1^2 - 1 = 1 - 1 = 0$

$Z_2 = 0^2 - 1 = 0 - 1 = -1$

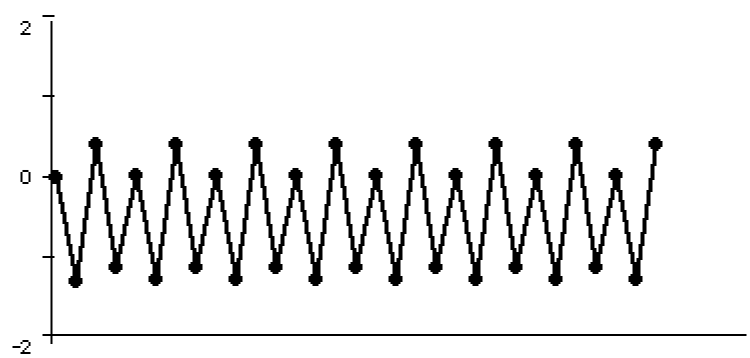
$Z_3 = -1^2 - 1 = 1 - 1 = 0$

Under the conditions z = 0 and c = -1 we see that the orbit flips back and forth between 0 and -1. Mathematicians would say that the behavior here is like a cycle between two points, or a “cycle of period two.” In both of these latter cases, the system does not head off to infinity but rather settles in around a fixed set of points, which is a very different kind of “behavior” than what we encountered in the first case.

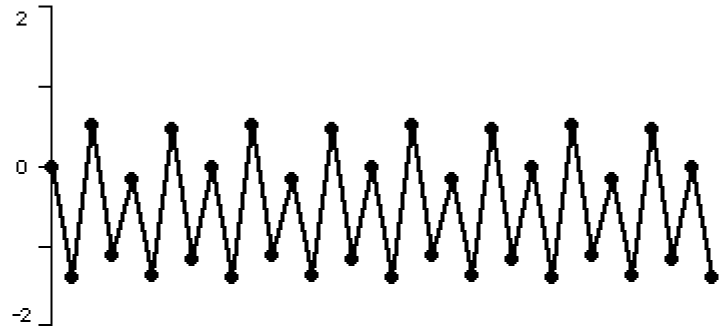
Indeed, if we were to play around with different values for C, we would witness a host of different behaviors. At z = 0 and C = -1.1 we get a cycle like this:



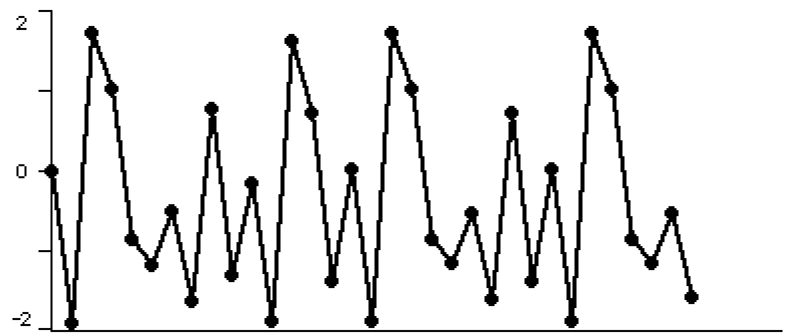
At C = -1.3 we get:



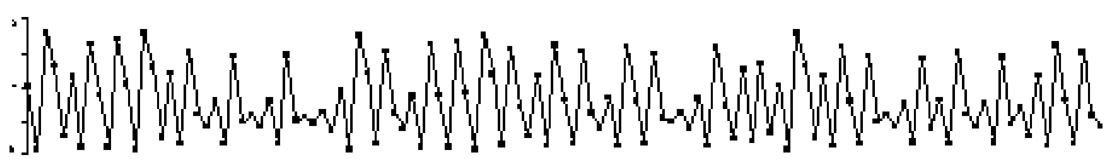
At C = -1.38 we get:



And at C = -1.9 the behavior looks like this:



In the cases above, the orbits settle upon a fixed number of points, two, three, five, etc. In some cases, the orbits do not settle upon a fixed number of points but are all over the place, as when C = -1.85:



In this case, mathematicians would describe this as "chaotic behavior." But note that in each of these cases, all of the results stay within a fixed boundary or orbit; in other words, the results do not head off toward infinity as they did in the first case. That distinction—between points whose iterated behavior head off toward infinity and those that do not—will become very important for the kinds of mathematical objects I want to describe below.

Indeed, were we to take all of those numbers for C and plot them on a number line, we could then locate different numbers on that line and note those numbers that, when plugged into the equation under the condition z = 0 then that C either goes off to infinity or huddles around an orbit, even if it is a chaotic orbit. On our number line, we would color or otherwise label the number 1 differently than we would 0 or -1 or -1.9. That is, we would want to identify those values for C where the number goes off to infinity and those that do not. Again, we are doing this for purposes that will become much, much more interesting in just a moment.

It should be clear that we would be able to identify a threshold after which our C number heads off toward infinity, and numbers on the other side of the threshold which cycle around fixed points. (I haven't calculated where that threshold is, but it probably hovers somewhere just after C = 0) Similarly, we would do the same kinds of iterations only instead of starting with z = 0 we could alter the value of z , say starting with z = 1, and see what results we might get.

All of the z and C numbers we have been using thus far are called "real numbers," the kinds of numbers we encounter in everyday life. But mathematicians have discovered (or invented, a distinction I want to return to later) other kinds of numbers. One mathematician in particular was interested in iterated functions, indeed in the same iterated function we have been using, only rather than using real numbers he plugged in "complex numbers." This is Gaston Julia, a French mathematician who, as you can see, was oddly-shaped. (His nose having been lost in the First World War, meaning he wore that odd piece of leather over his missing proboscis.)

You will soon learn that I am not disparaging Julia his physically odd shape, but am rather pointing to the odd shapes he discovered when using complex numbers in this iterated function.

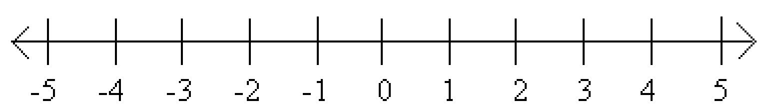
To understand what Julia had discovered, we must first explain what complex numbers are. A complex number is defined as a real number added to an "imaginary number," and is usually written in the following form:

$$a + bi$$

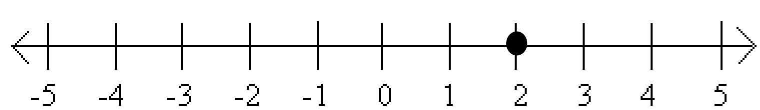
where a and b are both real numbers, and where *i* is an "imaginary number." You may recall from your high school algebra that an imaginary number is identified as the square root of -1. Mathematically, you cannot derive the square root of a negative number, but for certain kinds of mathematical operations using the square root of a negative number can nevertheless be valuable. So, mathematicians have "invented" this number and given it the symbol i and employ it in a variety of situations, including complex numbers.

So in a complex number, a + bi means that we take a real number (a) and add it to a real number (b) multiplied by i.

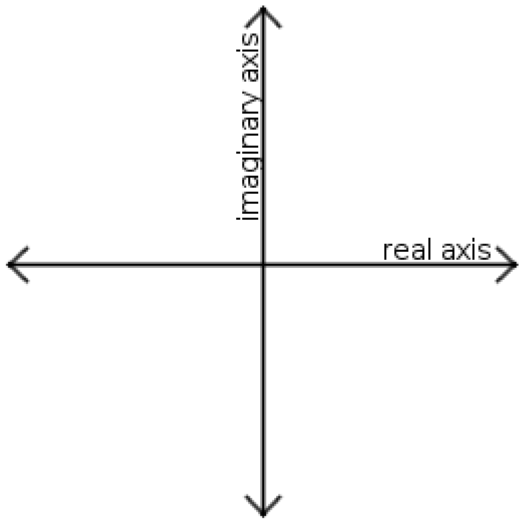
We can, of course, identify or plot any real number on a number line:



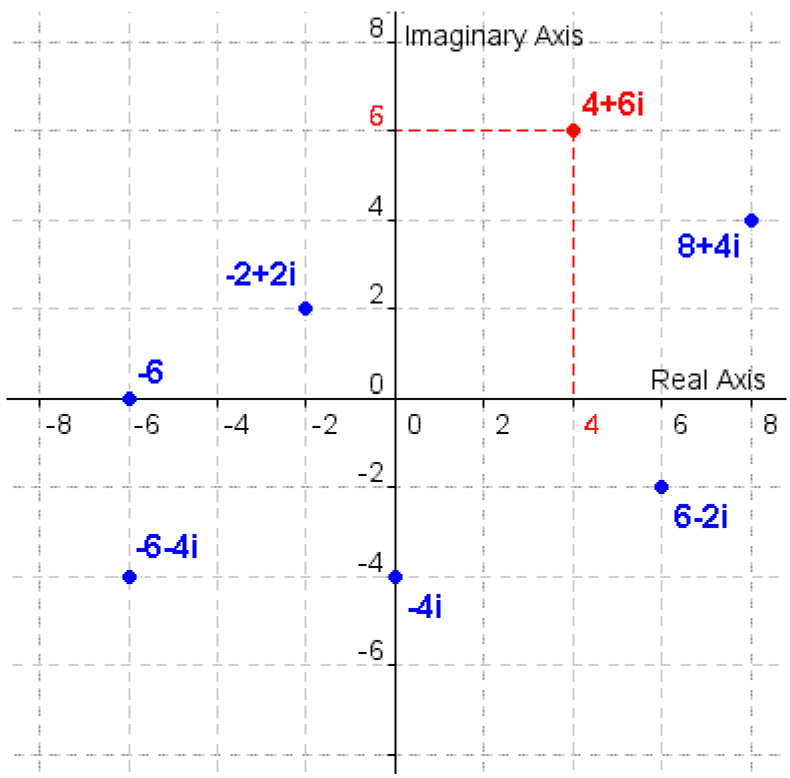
This is elementary school mathematics. When I want to locate the real number 2, I can easily plot this on a number line:



Because a complex number is made up of two parts, a one-dimensional line is insufficient for us to locate and plot it. Instead of a number line, complex numbers are plotted on the "complex plane."



The complex plane is made up of one axis that locates the a value in the complex number, and another axis that locates the bi portion of the complex number. So, if we wanted to plot a set of complex numbers, it would look like this:



The “complex plane” proves to be vitally important for the mathematical objects I want to highlight in this talk.

Back to Gaston Julia. Julia wanted to know what would happen when the z and C values in our iterated function were complex numbers. Julia set about calculating the behavior of our iterated function $z = z^2 + C$ by plugging a complex number for z then selecting another complex number for C. (Julia was undertaking a slightly different set of calculations than what we were engaged in previously. Where we started each iteration a z = 0, Julia was looking at lots of different values for z.) He then moved to another complex number as the starting point for Z but kept the same complex number for C. He then went to another z term, then another, in each case keeping the C constant the same complex number. I am not going to show you the calculations for these complex numbers (squaring complex numbers proves to be tricky, but for our purposes the calculation, and the results, are less important to us. Remember, what we want to know is what the behavior of the system is under each of these conditions.) Julia wanted to identify which of the complex numbers as z headed off for infinity and which ones settled into an orbit. On his complex plane, Julia colored those points that settled into orbits black and did not color or otherwise note the points that went off to infinity. He then plotted these results out on the complex plane, and here’s what he got:

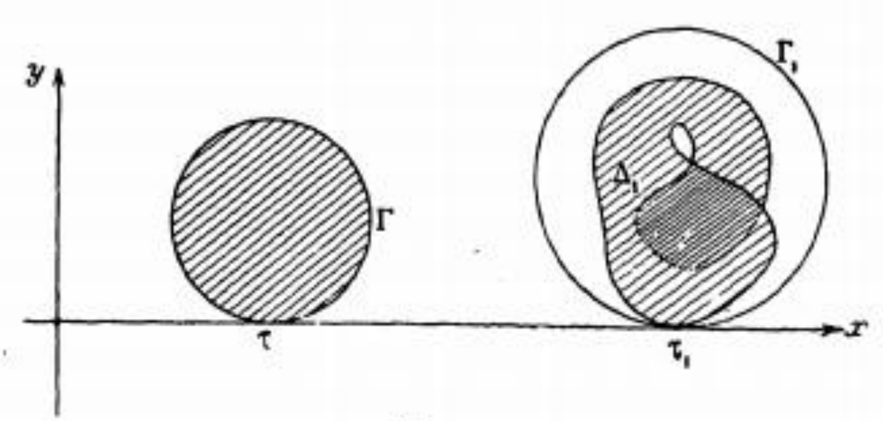
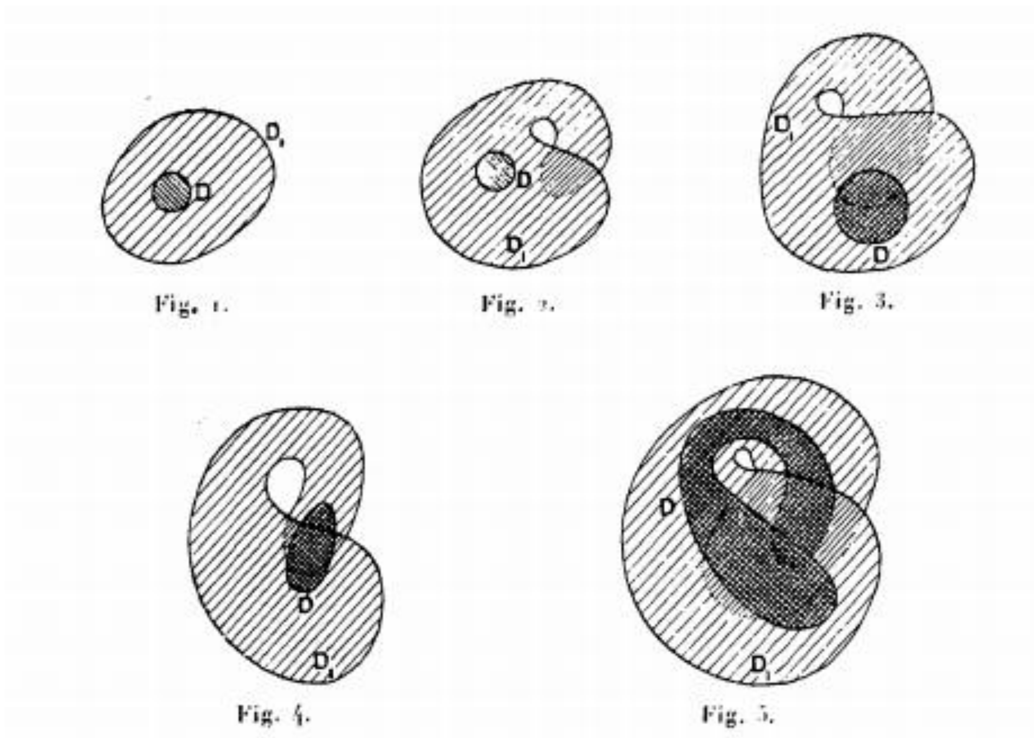


Fig. 10.

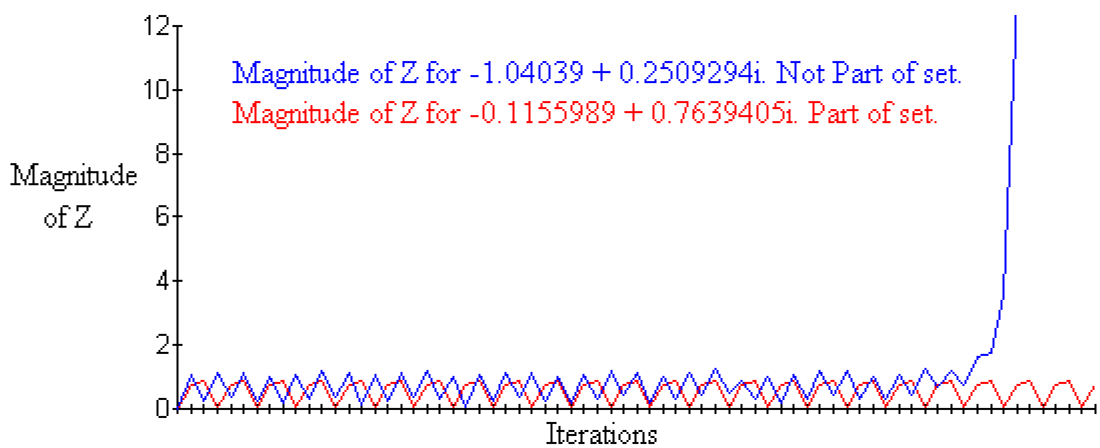


What you are looking at are the iterated results drawn out on the complex plane. Any dark spot in this diagram means that the complex number at that point (the z value in our iterated function) is not going off to infinity. Remember that Julia was keeping the C constant in each case. He did a number of these calculations using different values for C. In each case, he plotted each of the numbers on the complex plane, looking for those values that did not head off toward infinity. By this method, Julia produced a number of what we call "Julia Sets." A Julia Set is a set of those complex numbers that, in the iterated function $z = z^2 + C$, stay within a fixed orbit.

These shapes are…well, "misshaped." Mathematicians refer to these kinds of oddly shaped geometries as "monsters," in that they do not seem to conform to our expectations of Euclidian regularity and simplicity. What are we to make of these odd shapes? What can they tell us? What's the point of this whole exercise anyway? In 1918, Julia wrote a 200-page manifesto titled "Memoir on iterations of rational functions," which described his investigations of iterated functions using complex numbers on the complex plane. His work and his misshaped monsters drew a flurry of attention from mathematicians and the general public, but were then just as quickly forgotten.

Until the 1970s, when a mathematician at IBM, Benoit Mandelbrot, took up the investigation of these odd shapes. (Mandelbrot's uncle, himself a mathematician, had urged Mandelbrot to explore these now-forgotten monsters.) Mandelbrot worked at IBM (and Harvard University) and, in the 1970s, had access to computers and computing power not available to Julia. Julia was hand-calculating his results. Mandelbrot was able to calculate and iterate many more complex numbers than Julia would ever be able to calculate by hand. In addition, Mandelbrot was plotting out his results using computer graphics that were becoming more sophisticated and with finer resolution than had been previously seen.

Mandelbrot was using the same iterative function $z = z^2 + C$ that we have been using all along. Like the first set of examples I showed you, Mandelbrot was interested in iterations that started with z = 0. (Indeed, for the all of the results you will see, z always starts at 0. Remember, Julia was looking at lots of different values for z.) Like Julia, Mandelbrot was using complex numbers as C values. What Mandelbrot did was to take a complex number, set that number as C, iterate it through our function beginning with z = 0, and determining the behavior of the results.



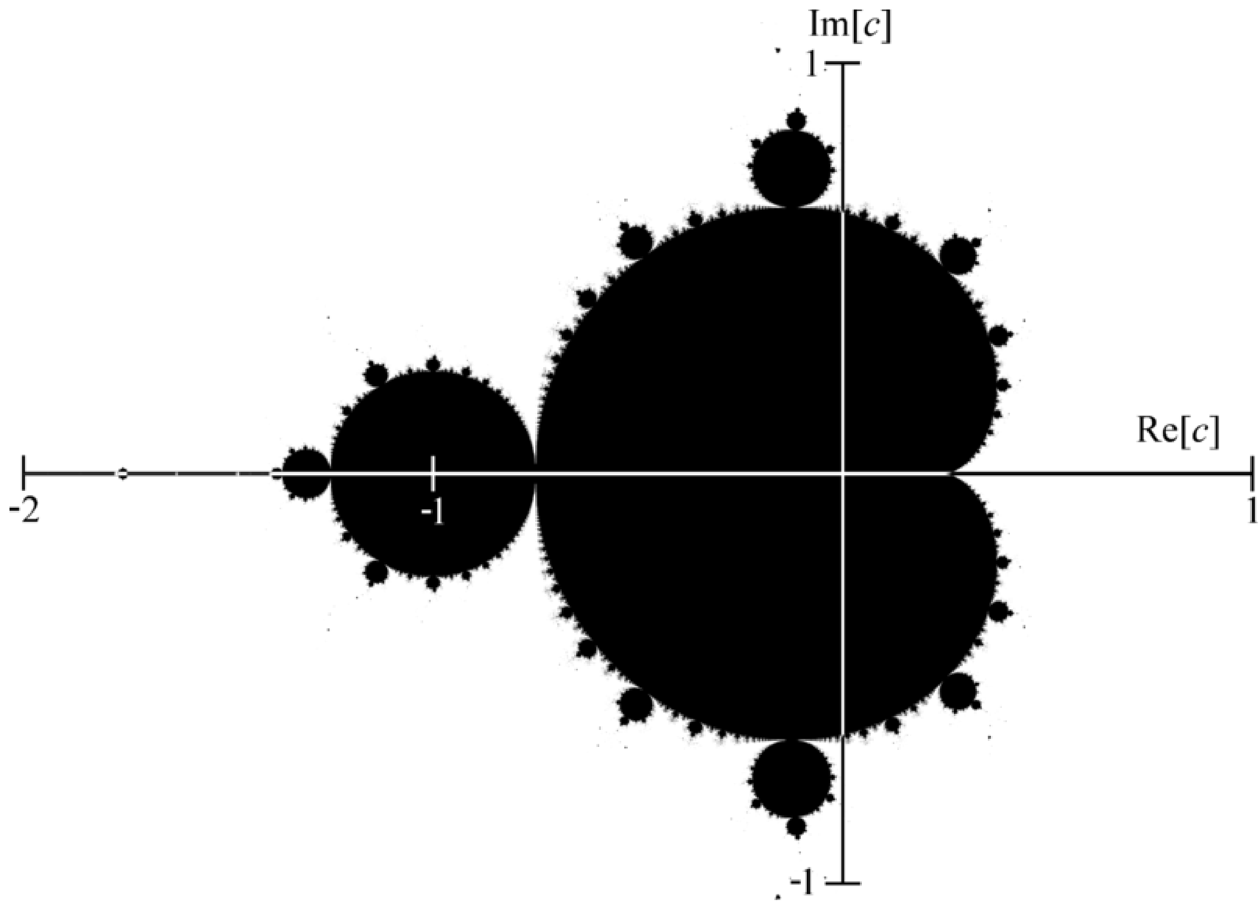
If the iterations yielded behaviors that settled into an orbit, he colored that point in. So, in the above case, when C = -0.1155989 + 0.7639405i, because the iterations stay in an orbit, that point would be colored black. When C = -1.04039 + 0.2509294i, the iterations head off toward infinity, and so that point was left blank when plotted on the complex plane. (Again, I'm not going to show you the calculations, because these are complicated. We are only interested in the behavior of the iterated results, and whether or not these head off to infinity or stay in orbit). Those complex numbers on the complex plane that settle into orbits when iterated at z = 0 were all colored black and are said to be contained within what became known as the Mandelbrot set (as in a set of complex numbers).

As I noted, Mandelbrot had access to computing power that Julia did not. Here is the result of the first set of plots, composed on a computer in 1978:

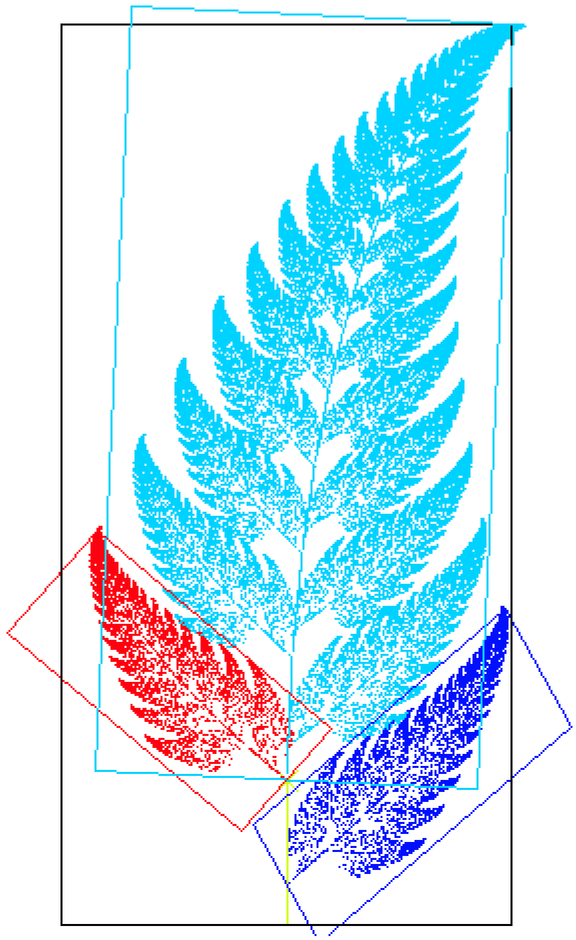
```
                                                     *
                                                   ****
                                                  ******
                                                   *****
                                              * *********
                                         *** ****************
                                           ******************** **
                                        *************************
                                       ****************************
                                     *******************************
                                     ******************************
                        * *****     ********************************
                       ***********  ********************************
                      ************* *******************************
                  ** ************** ******************************
****************************************************************
                  ** ************** ******************************
                      ************* *******************************
                       ***********  ********************************
                        * *****     ********************************
                                     ******************************
                                     *******************************
                                       ****************************
                                        *************************
                                           ******************** **
                                         *** ****************
                                              * *********
                                                   *****
                                                  ******
                                                   ****
                                                     *
```

Again, each of those points represent points on the complex plane that, when iterated in our function, settle into orbits and do not head off to infinity. The shape is as “misshapen” as those identified by Julia in that it does not look like a perfect circle or square or parabola. But there is clearly something else happening here, there is some sort of interesting pattern or underlying order that seems worthwhile to explore.

Indeed, as Mandelbrot employed more powerful computers with higher resolution graphics, it became clear that the results were more interesting than anyone could have imagined. Here is the Mandelbrot set plotted using better resolution graphics (with the complex plane included):

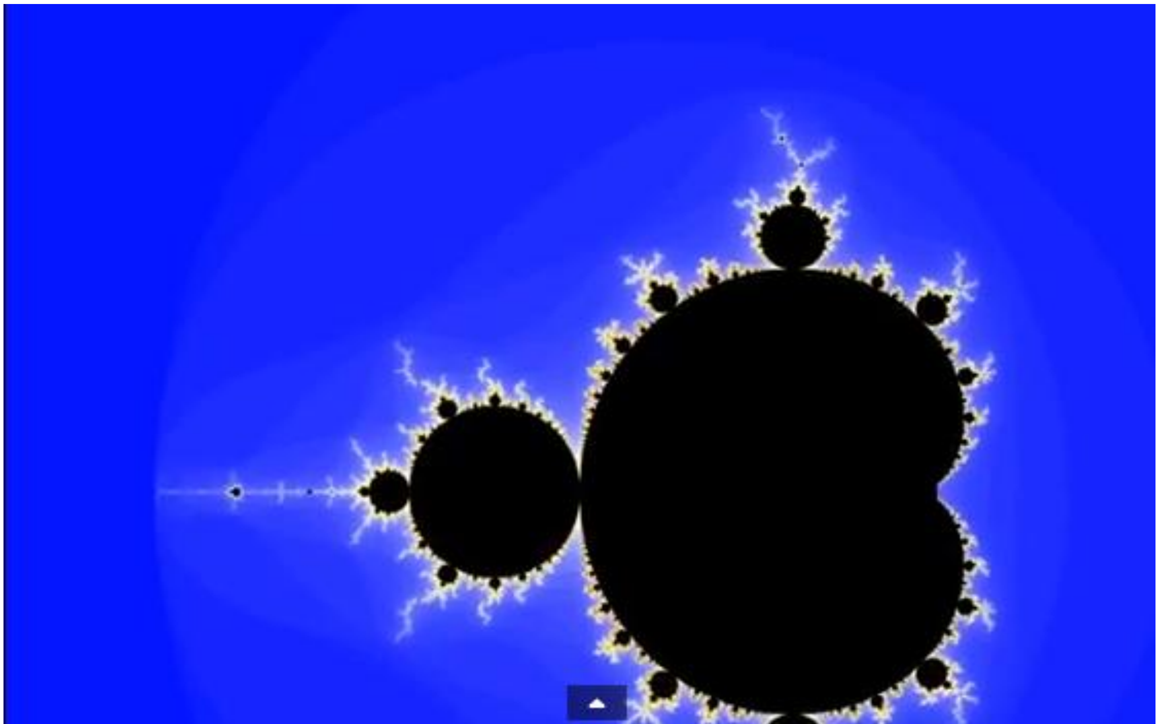


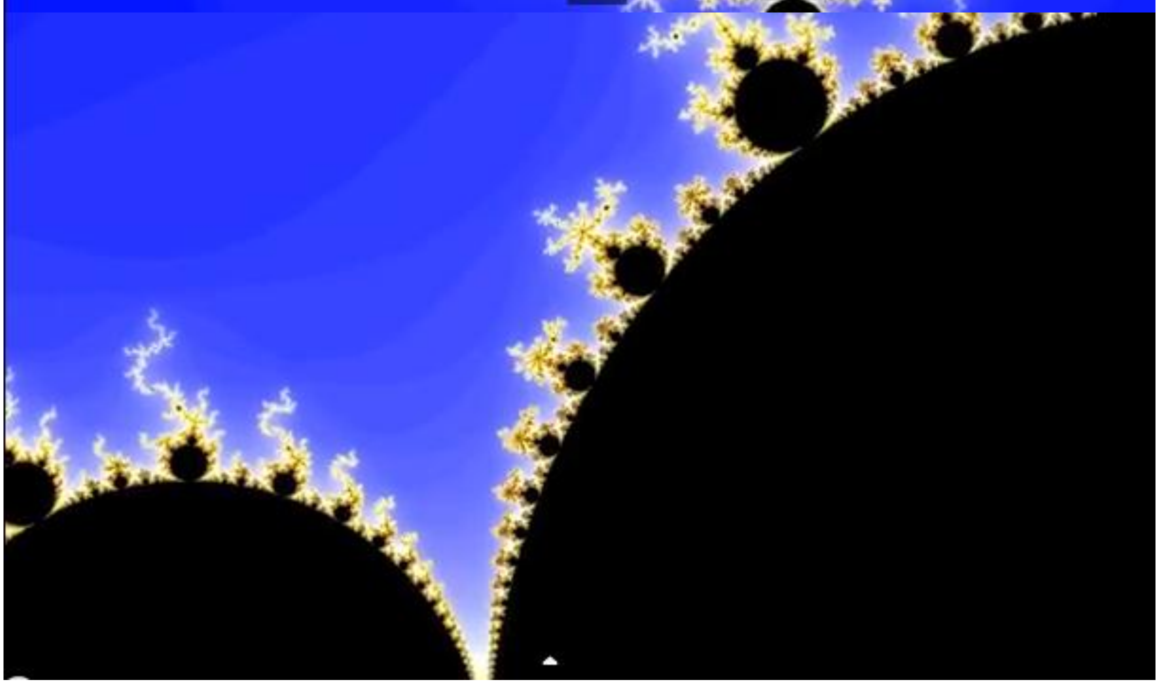
Under these conditions, some interesting behaviors become apparent. First of all, we note that the set of those complex numbers that settle into orbits adheres to a heart-shaped “bulb.” But what also becomes clear is that that bulb has “branches” that seem to replicate that heart shape. Indeed, if we were to zoom in on one of those branches, we would see a shape that looks very similar to the larger overall shape. This property is called “self-similarity at scale,” meaning that the same or similar shape can be found at whatever scale we might be located at in the object. Think of the branching pattern in a fern that looks very similar when we descend down into an individual branch, and then down even further at the level of individual fronds.

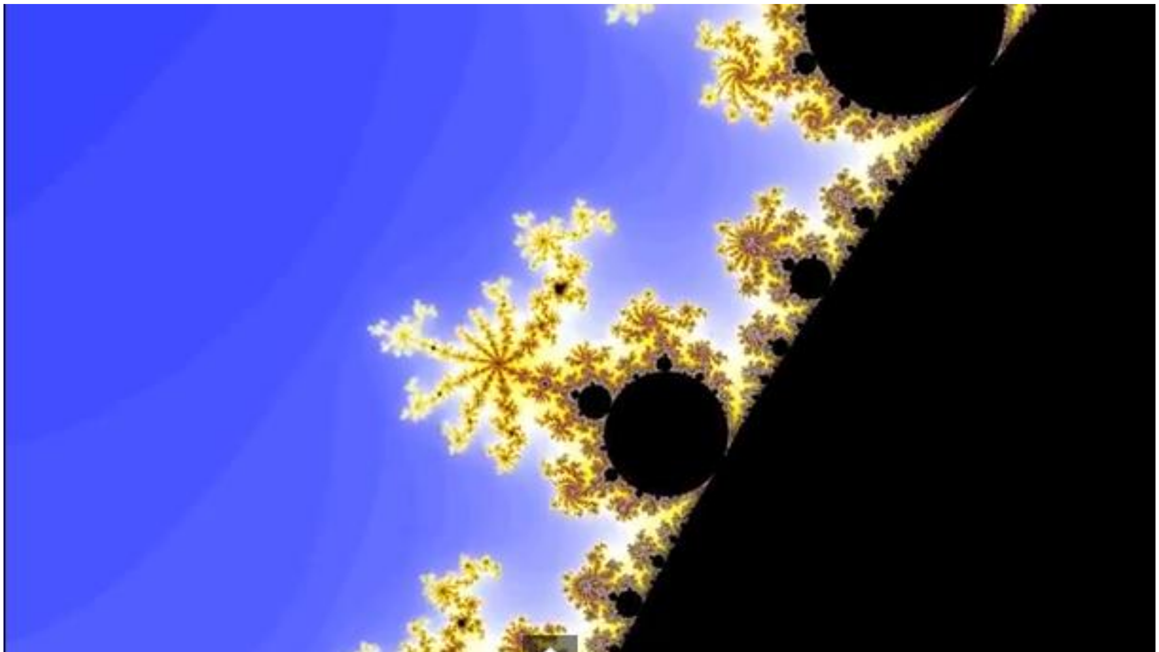


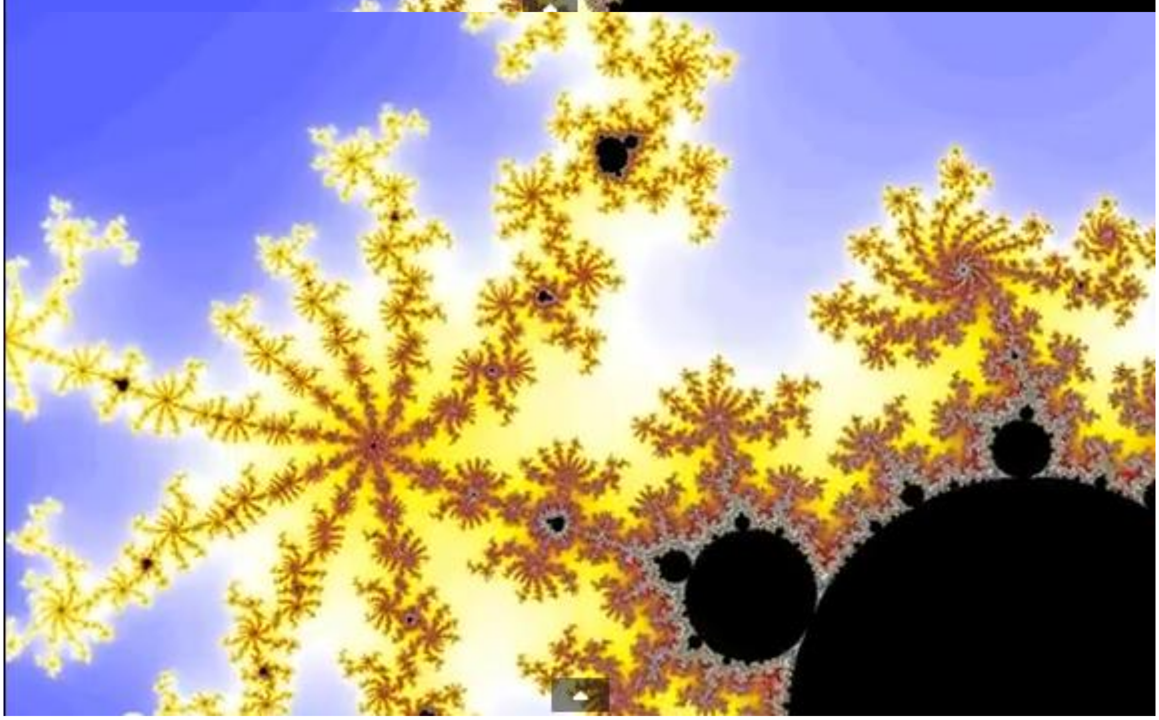
Or think about the self-similarity we might see in a stalk of cauliflower. This kind of self-similarity at scale is something we find in nature all the time, but very rarely in our Euclidean geometric formulations.

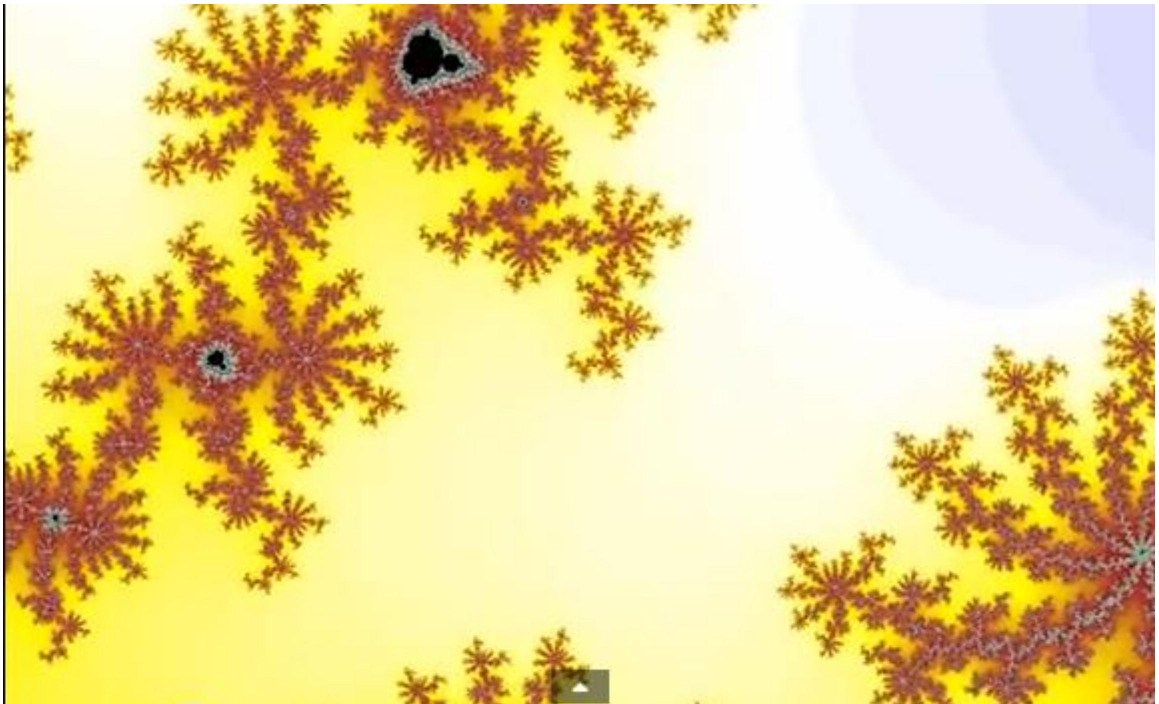
Up until this point, we have been ignoring those points that head off to infinity. But it turns out that plotting those complex numbers yields very interesting results. Using the kinds of computer graphics now available in the early 1980s, it was possible to plot out more numbers in an array of colors. We can take the step of looking at those C points that head off to infinity, and note especially the rate at which these numbers headed off to infinity. So, if the iterations rapidly headed off to infinity, we give it one color. The slower the rate, we use a different color. As I noted in my real number line example, it should be possible to locate a "threshold" between those points that settle in orbits and those points that head off toward infinity. It turns out that when you look for that "threshold" between those points in the Mandelbrot set and those outside the set, you get some of the most stunning vistas ever witnessed anywhere on the planet. Powerful computer graphics allow us to "zoom in" to deeper and deeper levels of the Mandelbrot set:

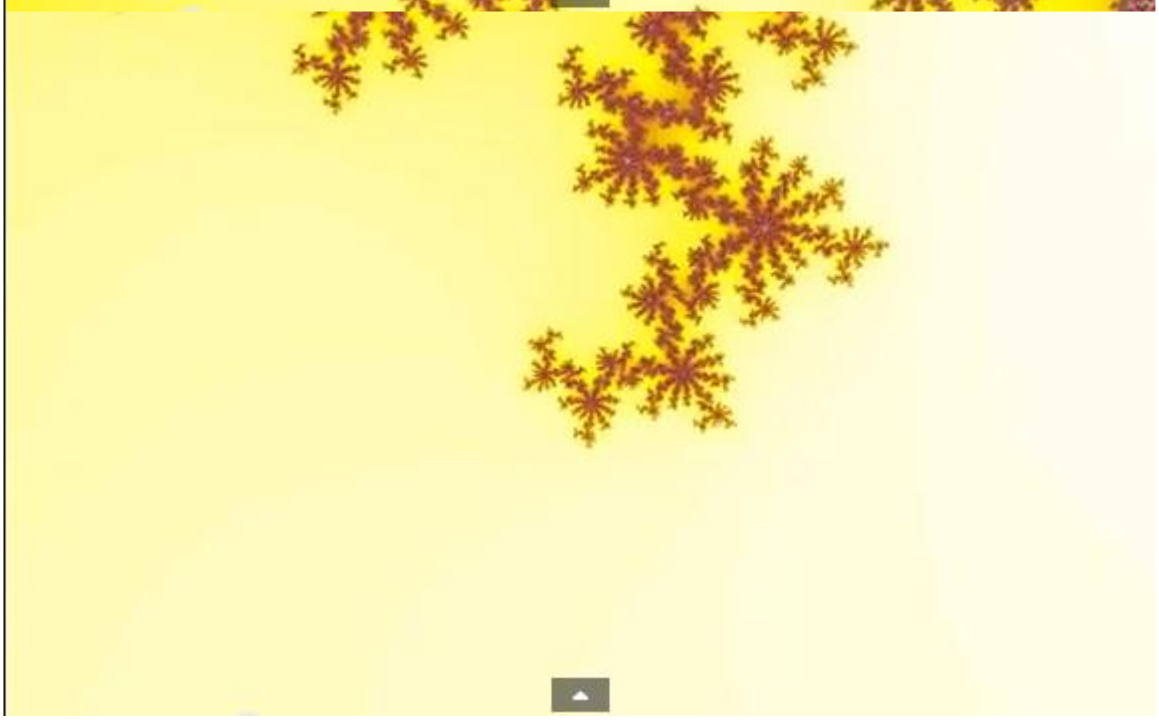


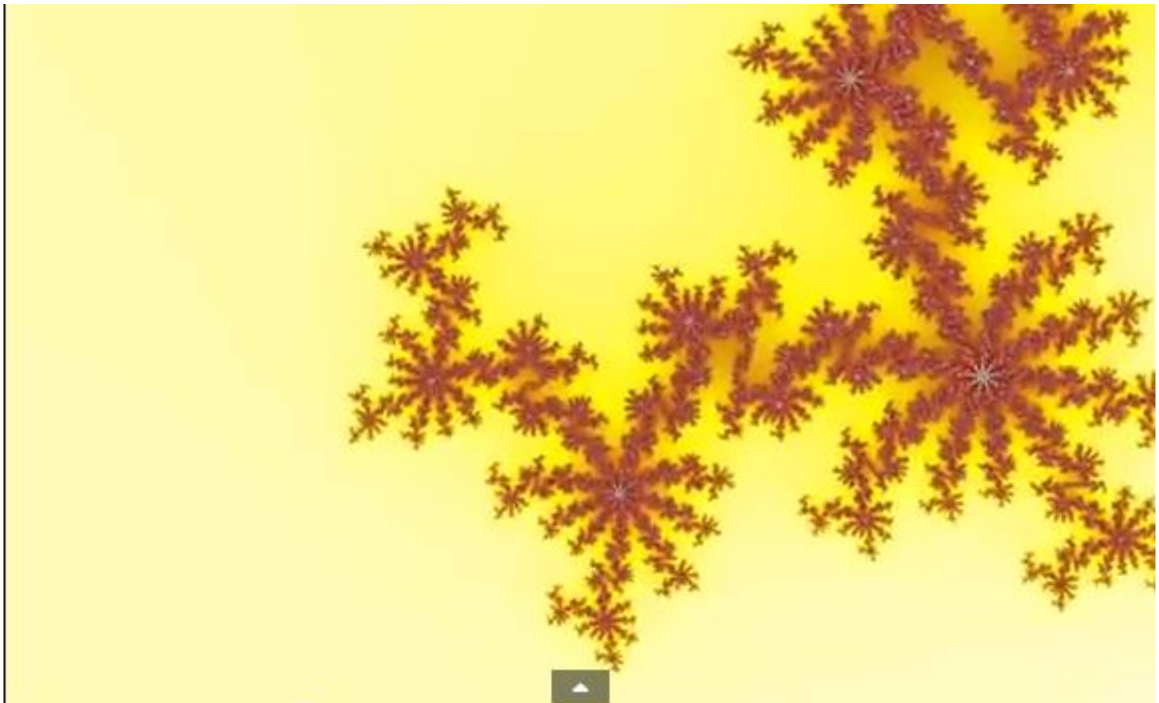


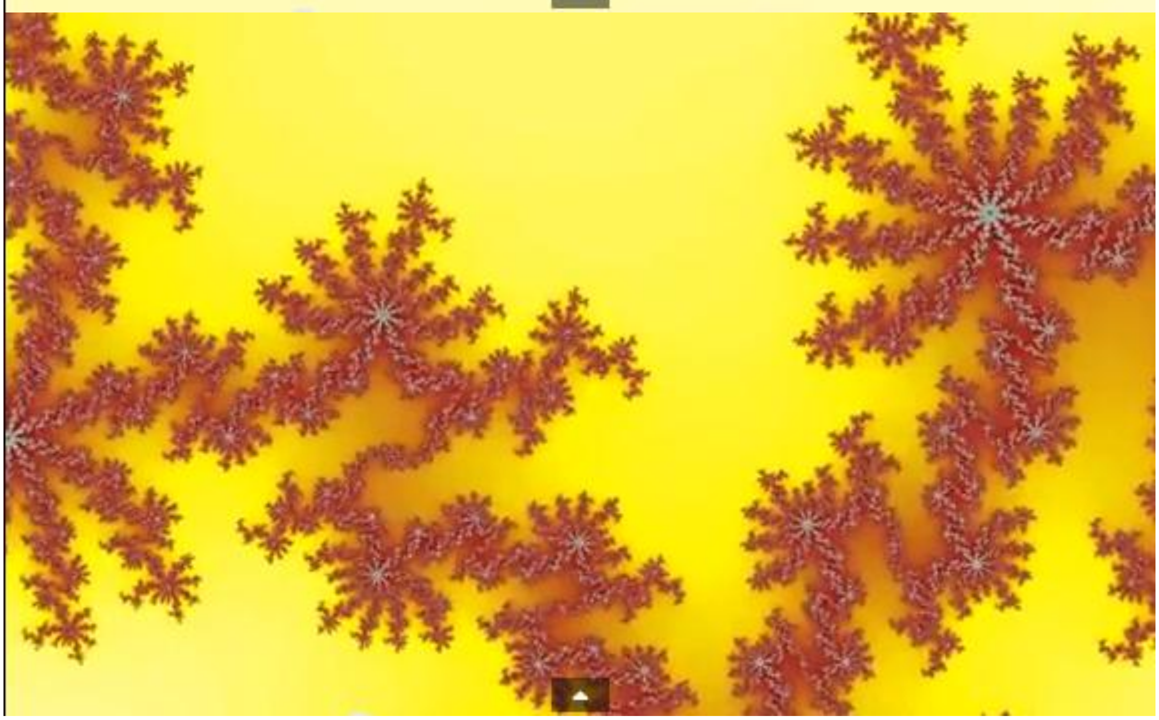


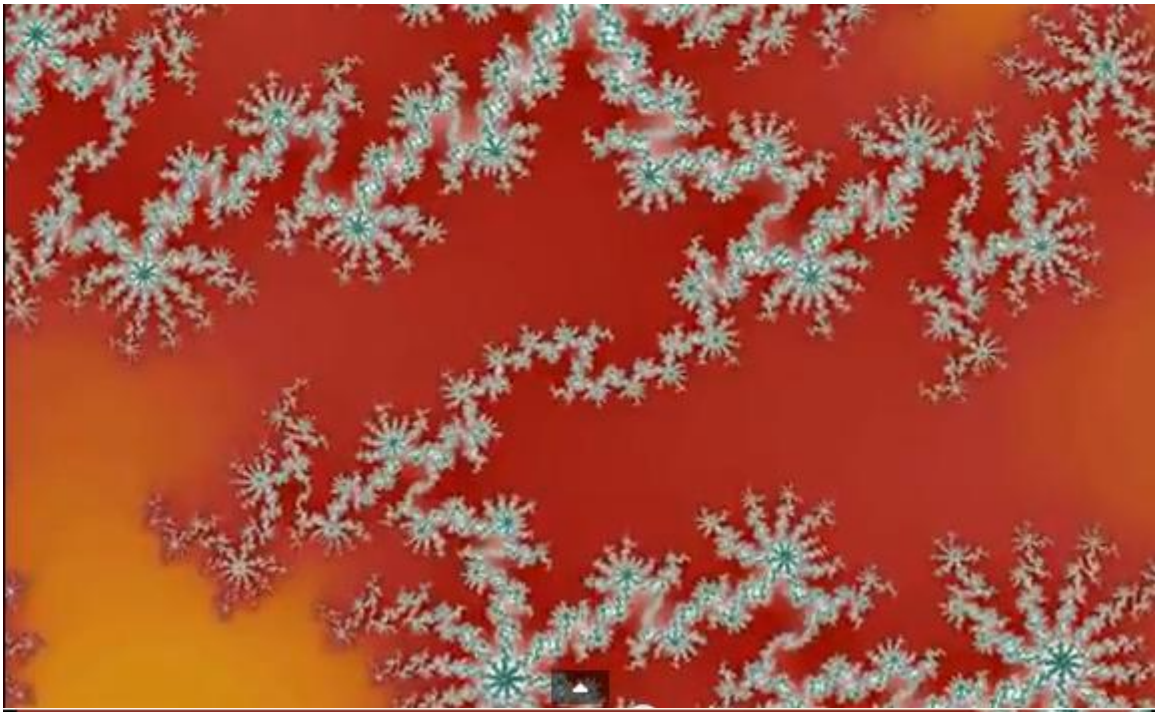


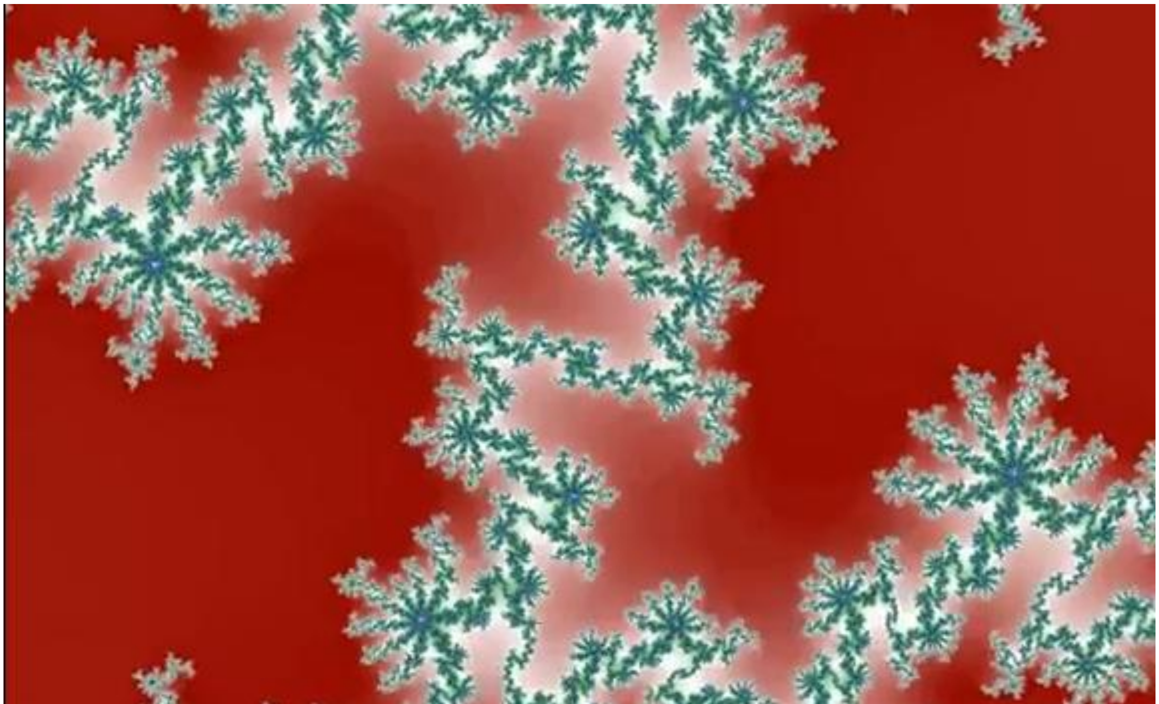
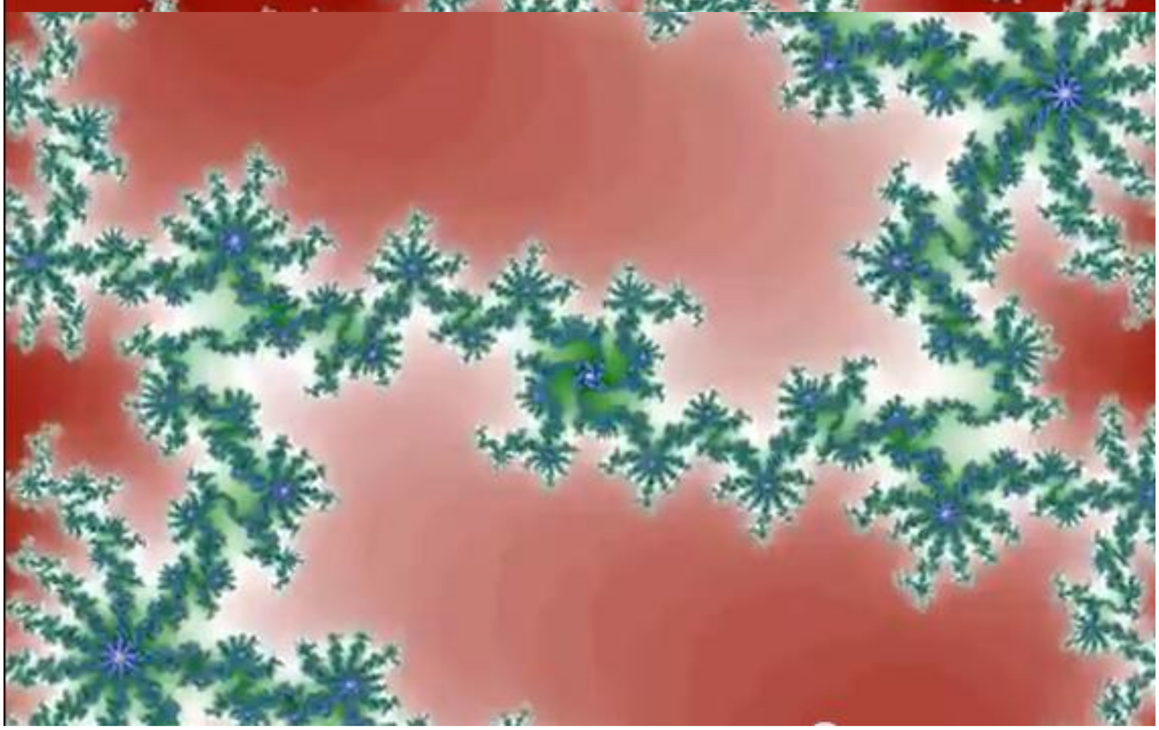


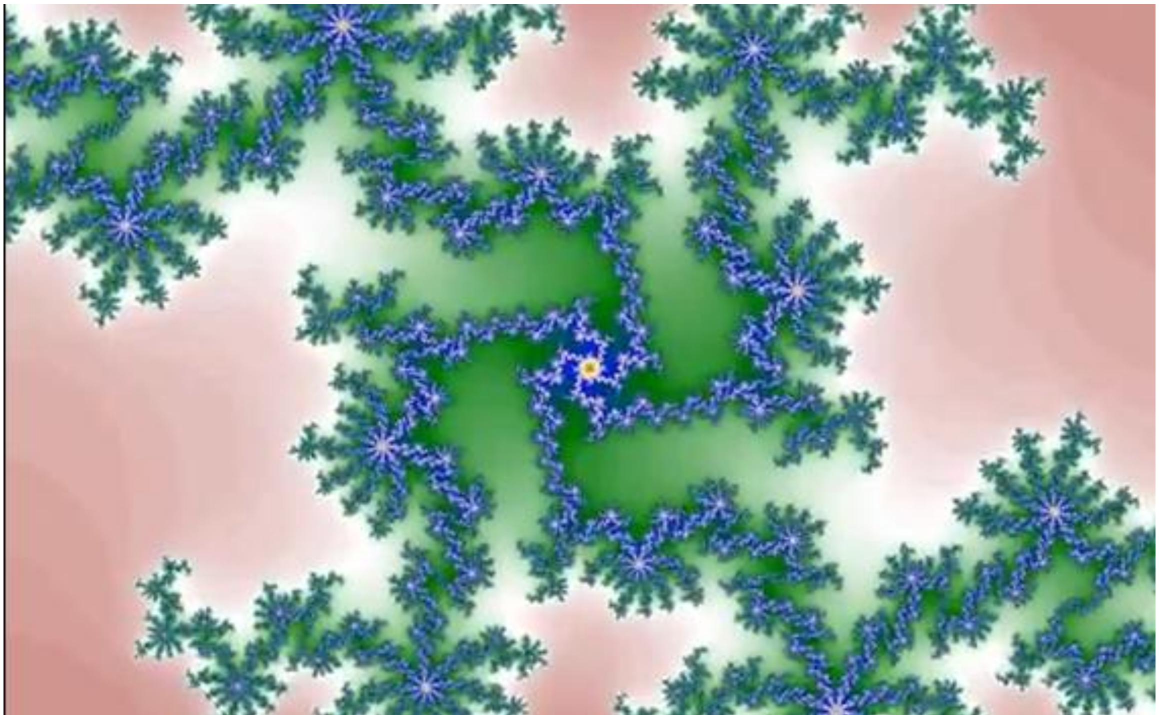


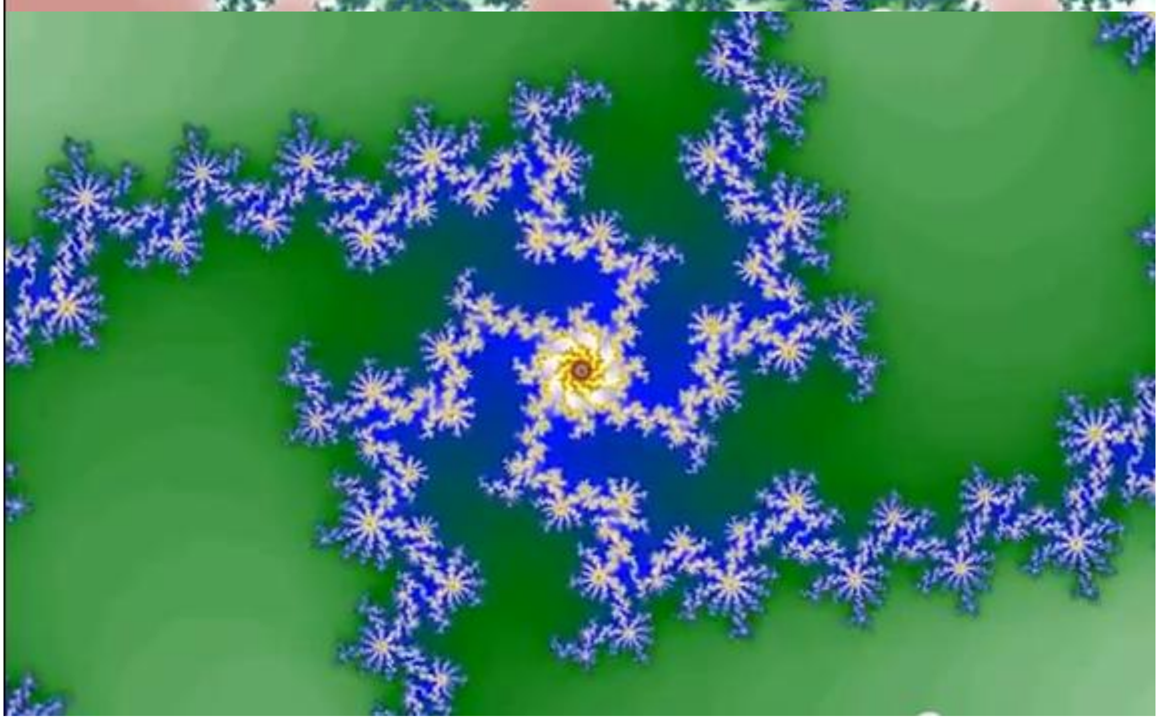


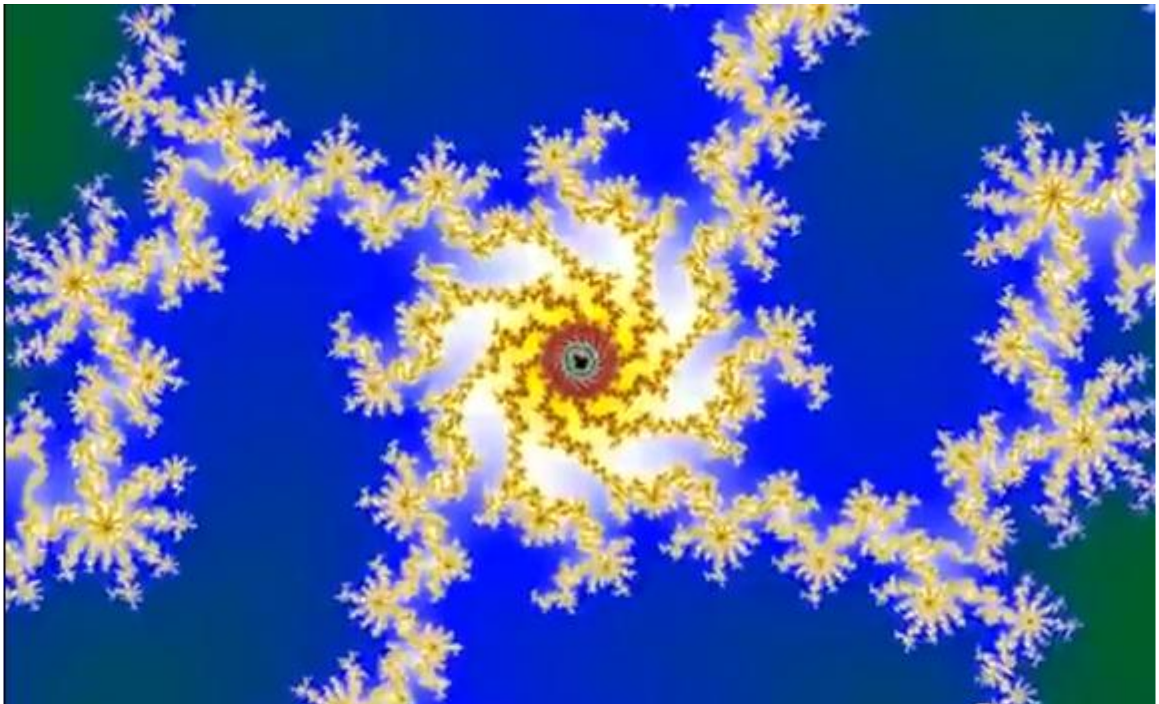


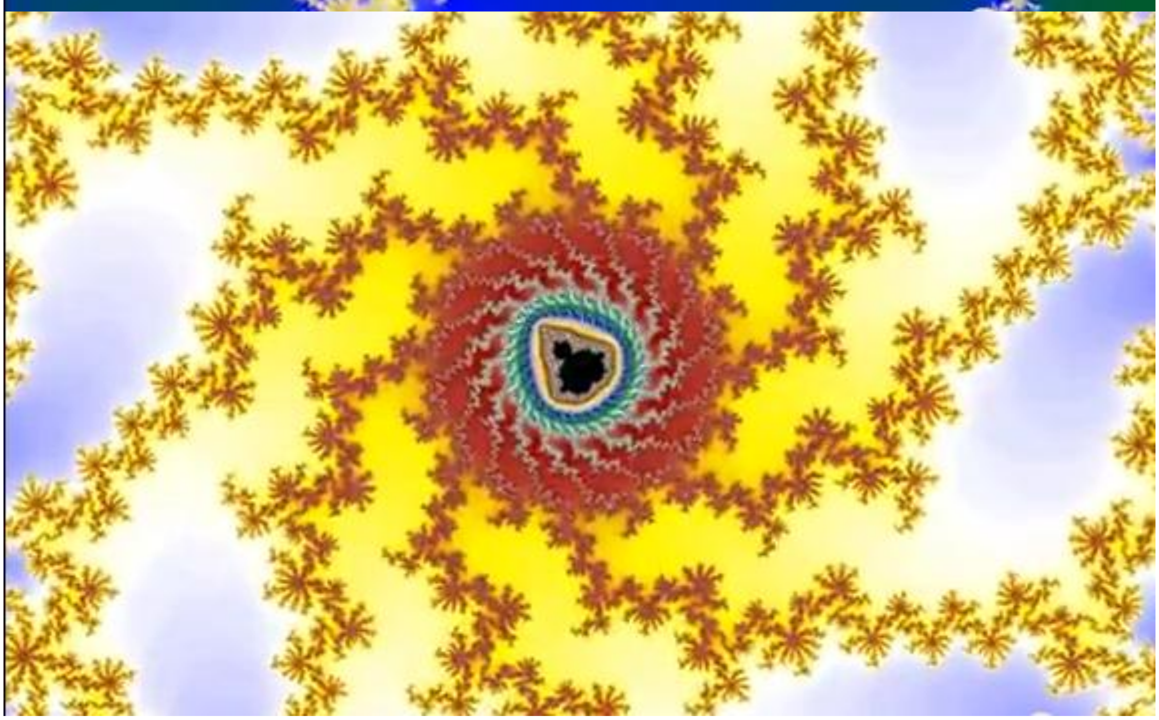


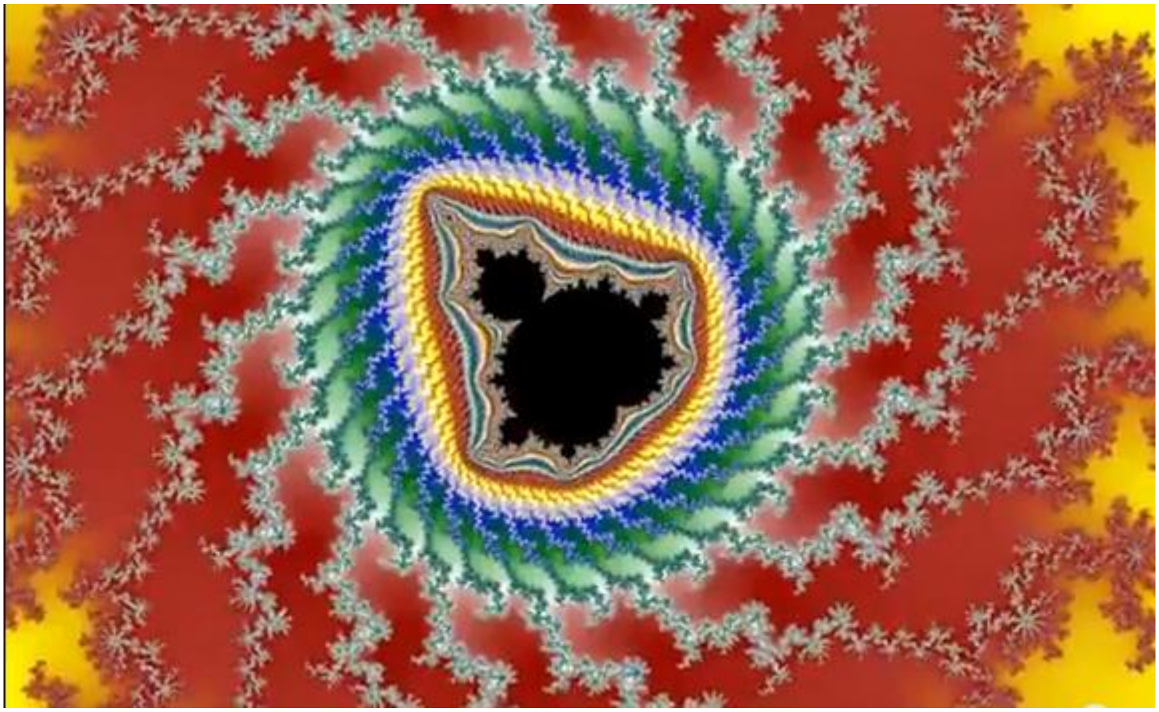







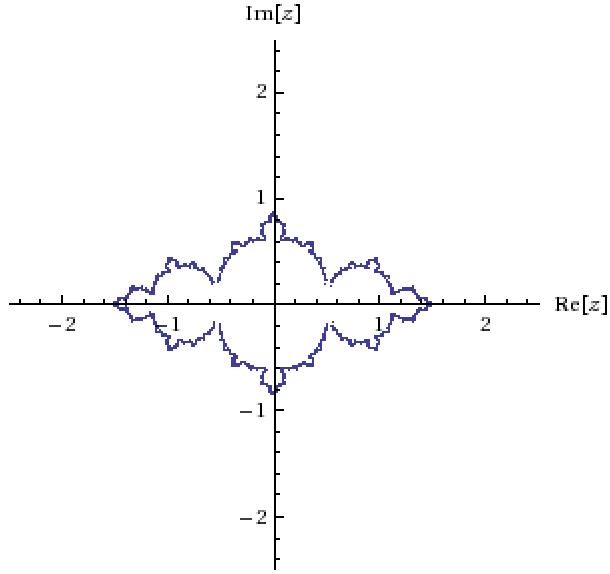






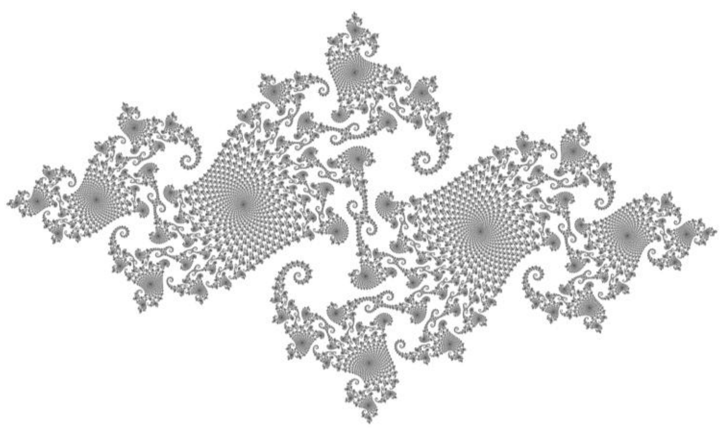
The mathematicians who first “explored” this area of the set referred to it as “Seahorse Valley,” obviously named after the interesting patterns produced there. Indeed, using complex computer graphics, smaller and smaller regions of the Mandelbrot set could be explored, in the same way we might explore the ocean depths. Those explorers were afforded the privileged of “naming” such regions of the Set, the way other explorers have named the landforms they discovered. Even at the great depths of Seahorse Valley, note that the “heart-shaped bulb” appears. Indeed, that shape appears all over the Mandelbrot set, turning up unexpectedly at whatever depth you wish to consider.

By the way, those same computer graphics can now be used to plot the Julia Sets that Gaston Julia could not easily calculate or plot out in 1918. When we do so, here’s what we get:

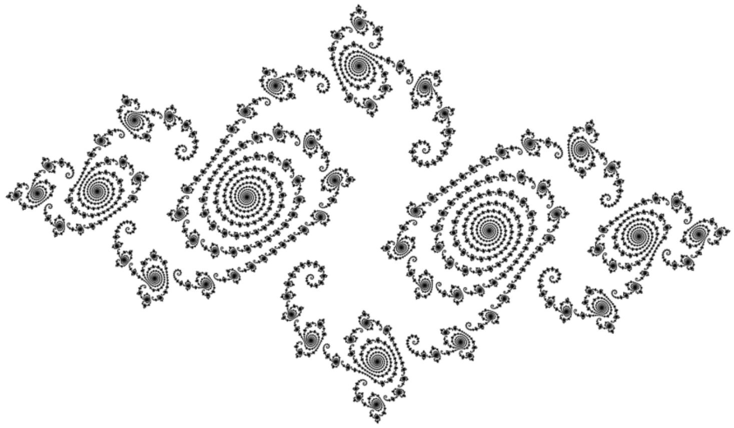


C = -0.765 + 0.003i

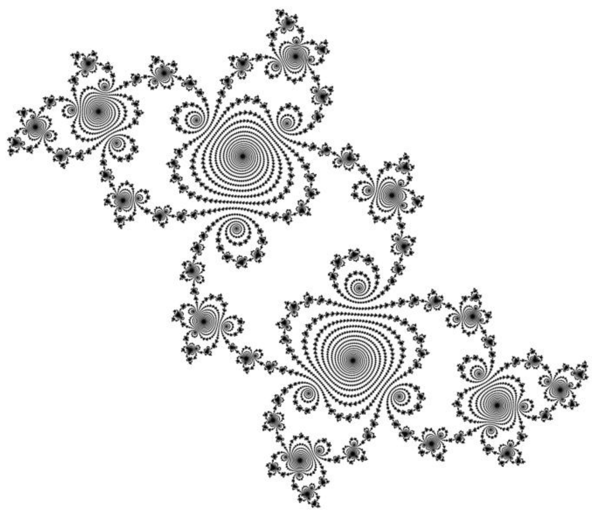
There are many Julia Sets, each plotting those points on the complex plane that remain in orbit under the iterated function $z = z^2 + C$ When we change the C value, we produce a new Julia Set.



c=-0.74543+0.11301i



c= -0.75+0.11i

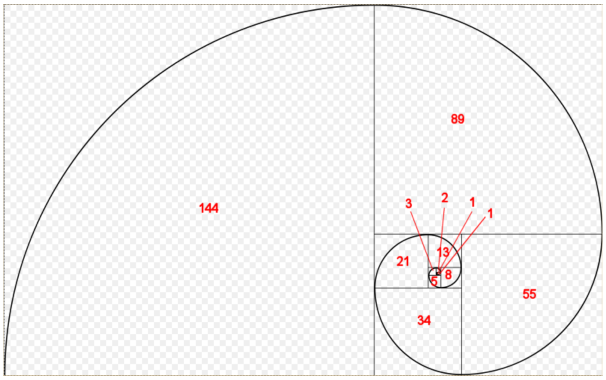


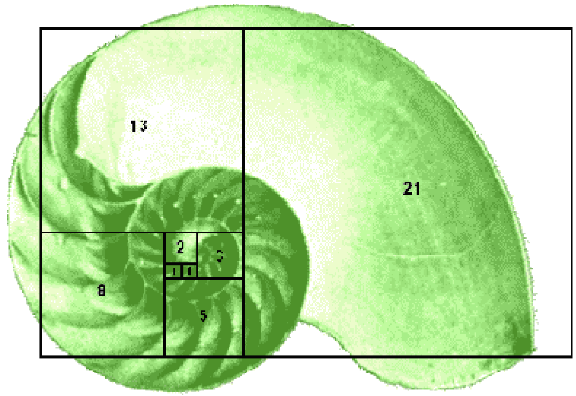
c=-0.1+0.651i

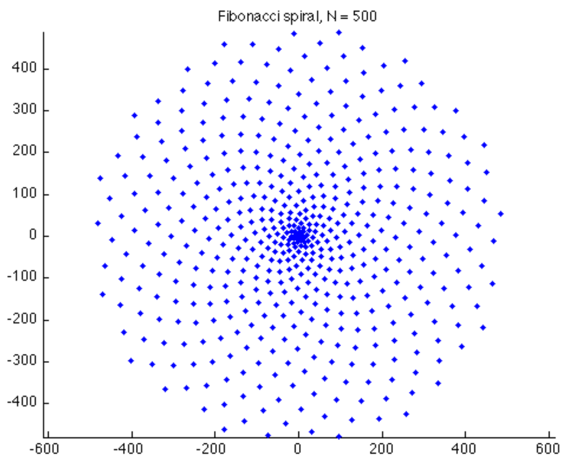
Julia's monsters seem less monstrous when we plot them in this fashion. I cannot emphasize enough that without the development of computer graphics we probably never would have unleashed the beauty and complexity of Julia Sets and the Mandelbrot set. We would have never been able to see—and thereby explore--these mathematical objects.

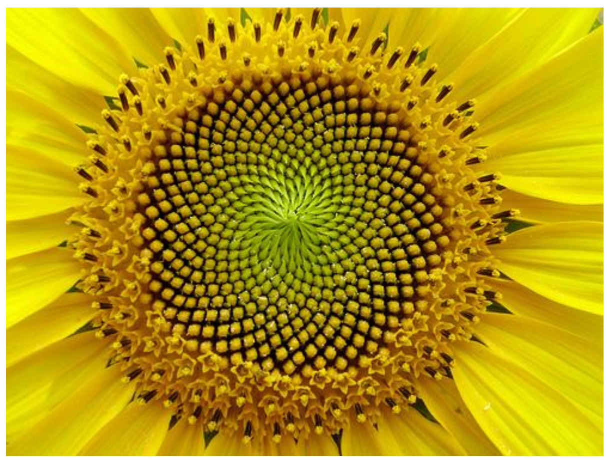
What had Mandelbrot discovered? (Or was it "created?") Galileo is supposed to have said "The Book of Nature is written in the language of mathematics," and indeed we often assume that mathematics is useful (beyond calculating sums) as a way for us to better understand the natural world. The Fibonacci Sequence is interesting to us because that ratio of numbers

(1,1,2,3,5,8,13,21) has a specific relationship to the Golden Mean, and when arranged geometrically very well describes certain natural forms, like nautilus shells and sunflowers.

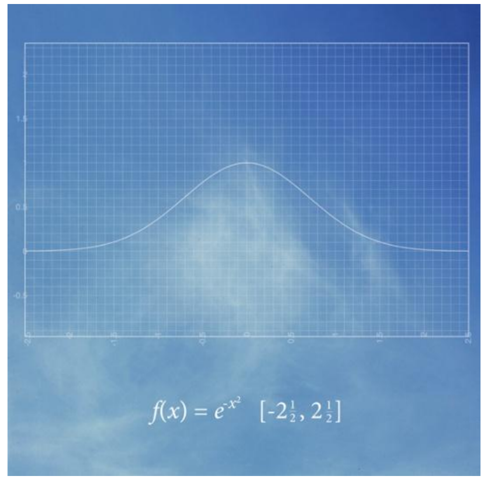








The photographer (and mathematician) Nikki Graziano displays this impulse to see mathematical forms in the world around us. But note how Graziano's equations have "smoothed out" the scenes. That is her equations do not align to the shapes she has photographed; the equations are approximations of those shapes.



The Mandelbrot set does not appear to map onto anything in nature, but nevertheless looks very “naturalistic.” Indeed, for his part Mandelbrot claimed to have discovered the mathematical principles of what he termed “roughness.” Some of the mathematics behind the Mandelbrot set was immediately grasped by computer graphics people, as a way to render more realistic and natural looking graphics for computer games. Mandelbrot claimed to have developed mathematics that more closely maps the rough contours of the natural world that equations such as Graziano’s smooth away.

But my own sense is that the Mandelbrot set does not replicate anything specific in nature, and is fascinating…simply because it is a fascinating object. I compare such mathematical objects to poems or music. Jim appreciates poetry for its beauty, Arnett finds complexity in music. Mathematical objects like the Mandelbrot set are analogous. Was the Set conjured in Mandelbrot’s imagination like a poem or musical composition? Or had he discovered a principle of the universe, like the complexities of pi, an object that exists somewhere between the natural world and the world of our imagination? Why do numbers behave the way they do? Are they our invention, or part of the substance of the universe? Mathematical objects seem to hover somewhere between, which is why they fascinate me so much. Mathematics is the science of patterns; it is not mere calculation. The question I have been asking myself for years is “what is the meaning of these patterns?” Are the patterns of number themselves a part of the universe, just part of a universe of abstractions that we access not by picking up shells or sunflowers but by exploring our imaginations. The patterns of mathematics reside in the imagination, and form their own kind of universe.

Mathematicians have described the Mandelbrot set as the most complex mathematical object in the universe. And remember, it is derived from the simplest of equations:

$$z = z^2 + C$$